Estimating the Covariance Matrix for Portfolio Optimization

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Abstract

We discuss the estimation of the covariance matrix of stock returns for portfolio optimization and show that for constructing the global minimum variance portfolio (GMVP), there is no statistically-significant gain from using more sophisticated shrinkage estimators instead of simpler portfolios of estimators. We introduce a new "two block estimator," which produces—in an unconstrained optimization—a positive GMVP, that can be found analytically and that is sensitive to even small changes in the covariance matrix. For constructing the GMVP, an example of our new estimator performs at least as well as a combination of imposing the short sale constraints and using the sample matrix.

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Estimating the Covariance Matrix for Portfolio Optimization

1. Introduction

The computational aspects of finding efficient portfolios have been a concern of the finance profession since the seminal work of Markowitz (1952, 1959). While the mathematics of efficient portfolios is relatively simple, the traditional practical implementation of the theory often leads to questionable results—in many cases the covariance matrix is not invertible, and in other cases the "optimal" portfolios have very large short-sale positions.

Essentially there are two approaches to deal with the problematic implementation results. The first one is the "theoretical approach," in which theoretical aspects and assumptions regarding portfolio optimization are re-examined. We will not follow that approach in this paper. The second approach is the "implementation approach," which mostly stems from the fact that the two main elements of Markowitz's Mean-Variance (MV) theory—the expected stock returns vector and the covariance matrix of the stock returns—are unknown, and thus must be estimated. In this paper we focus on the estimation of the covariance matrix.[1]

The estimation of the covariance matrix, like any other estimation process, contains an error. When discussing this error, it is common to distinguish between estimation error and specification error. The estimation error occurs when there are not enough degrees of freedom per estimated parameter, or in other words when the number of observations in the sample is not big enough compared to the number of the estimated parameters. The specification error occurs,

[1] A literature that stems from Black-Litterman (1991) discusses estimation of the expected returns, but abstracts from the estimation of the covariance matrix.

when some form of structure is imposed on the model that is being used in the estimation process, and therefore the estimator becomes too specific in comparison with reality.

The traditional and probably the most intuitive estimator of the covariance matrix is the sample covariance matrix based on historical monthly return data (henceforth—the sample matrix). However, as Pafka et al. (2004) state, this estimator often suffers from the "curse of dimensions": In many cases the length of the stock returns' time series used for estimation (*T*) is not big enough compared to the number of stocks one wishes to consider (*N*). As a result, the obtained estimated covariance matrix is ill conditioned. Typically, an ill conditioned covariance matrix exhibits implausibly large off-diagonal elements. Michaud (1989) points out that inverting such a matrix (as required by the MV theory) amplifies the estimation error tremendously.[2] Furthermore, when *N* is bigger than *T*, the sample covariance matrix is even not invertible at all.[3]

The literature, which deals with methods to improve the estimation of the covariance matrix, is too extensive to survey here. Therefore, we restrict our discussion to estimators that comply with the following three assumptions: 1. Stock returns are independent and identically distributed (iid). 2. Historical monthly return data should be used in the estimation process. 3. Sample variances are good estimators of the stock variances.[4] In recent years several studies

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[2] "Implausible" is, of course, a value of judgment. For a discussion regarding the condition number of a covariance matrix, see for example Ahn et al. (2005).

[3] The proof why the covariance matrix is not invertible when *N is* bigger than *T* can be found in Ledoit and Wolf (2003).

[4] That is, the diagonal elements of all the estimators presented throughout this paper are the same as the diagonal elements of the sample matrix.

have concentrated on such estimators. Basically, most of these studies stem from a fundamental principle of statistical theory—there exists a tradeoff between the estimation error and the specification error. Hence, in order to develop an improved estimator, the huge estimation error of the sample matrix must be reduced without creating too much specification error instead. Combining the findings of the studies of Chan et al. (1999), Bengtsson and Holst (2002), Jagannathan and Ma (2003), Ledoit and Wolf (2003), Ledoit and Wolf (2004b) and Wolf (2004) reveals that the best estimators of that type are the shrinkage estimators and the portfolios of estimators.

The roots of the shrinkage method in statistics are not related to covariance estimation and can be found in the seminal work of Stein (1955).[5] Roughly speaking, in our context, a shrinkage estimator is an optimally weighted average of the sample matrix with an invertible covariance matrix estimator on which quite a lot of structure is imposed and whose diagonal elements are the sample variances.[6] We use the term "optimal," because one derives the proportions of the two estimators in the weighted average by minimizing the quadratic risk (of error) function of the combined estimator. The optimal proportions guarantee the reduction of the huge estimation error of the sample matrix without creating instead too much specification error (which is related to the second estimator in the weighted average). The off-diagonal elements of the shrinkage estimator are moderate (or shrunk) compared to the large off-diagonal elements of the sample matrix. The variance elements in the diagonal are kept untouched.

[5] A beautiful description of Stein's work is brought in Efron and Morris (1977).

[6] It is straightforward to show that a weighted average of two matrices, one of which is invertible, is also invertible. Thus, the shrinkage estimator is always invertible.

Jagannathan and Ma (2000) use the concept of a portfolio of covariance estimators. A portfolio of estimators is an estimator consisting of an equally weighted average of the sample matrix and several other estimators of the covariance matrix whose diagonal elements are the sample variances and at least one of them is invertible. This concept has also been adopted by Bengtsson and Holst (2002); it is based on the logic that estimators based on different assumptions make errors in different directions. The portfolio of estimators diversifies among these errors and they hopefully cancel out. In essence, the portfolio approach builds on the tradeoff between estimation and specification error. By averaging the sample matrix (which suffers from much estimation error) with other estimators whose primary error is specification error an improved covariance matrix can be obtained.

Both the shrinkage estimators and the portfolios of estimators which appear in the literature have been shown to perform substantially better than the sample matrix. However, the shrinkage estimators are more complex than the portfolios of estimators, at least in their theoretical derivation. While a portfolio of estimators is simply derived by using an equally weighted average, the derivation of a shrinkage estimator involves solving a minimum problem for finding the proportions in the weighted average and estimating these proportions, as they depend on some unknown parameters.[7] In this paper we check whether, in terms of performance, there is any gain from using the more sophisticated shrinkage methods. We do this by running a performance contest, which is based on a "horse race" between several shrinkage estimators and portfolios of estimators. We use the ex-post global minimum variance portfolio (GMVP) as our

[7] For an example of the proportions estimator, see Ledoit and Wolf (2003). On the face of it, the shrinkage estimators are also more complex in their computational derivation. However, by using the Matlab program, following Ledoit and Wolf, computing them is relatively easy.

betterment criterion. We show that all the estimators perform within the same range, and that it is actually impossible to claim that one of them is the better than the other. Our conclusion is that one can use the simpler estimators rather than the more complicated estimators. That is, there is no real need to use the shrinkage estimators and instead one can simply use the portfolios of estimators.

A significant drawback of the shrinkage estimators and the portfolios of estimators is that they generate minimum variance portfolios incorporating significant short sale positions. Short selling is a significant implemental problem in portfolio computations: It is widely prohibited (mutual funds, for example, are not allowed to short sell) and many individual investors find short selling onerous or impossible. Therefore, to the extent that short sales are indeed considered an undesirable feature of portfolio optimization, there is some interest in finding an estimator of the covariance matrix that performs substantially better than the sample matrix and produces positive efficient portfolios.

Probably the most intuitive way to obtain positive portfolios is to add, to the portfolio selection problem, constraints that prevent the portfolio weights from being negative (henceforth—short sale constraints), no matter which covariance matrix estimator is used. Jagannathan and Ma (2003) show analytically that imposing such constraints can be interpreted as a means of shrinking.[8] In order to check empirically whether this way can also produce estimators that perform substantially better than the sample matrix when the short sale

[8] They show that constructing the constrained GMVP from the sample matrix is equivalent to constructing the unconstrained GMVP from a shrunk covariance matrix. Because the shrinkage operation is a function of the Kuhn-Tucker conditions for the constrained maximization, the Jagannathan-Ma theorem does not characterize the constrained solution.

constraints are not imposed, we run a new “horse race”—this time imposing the short sale constraints, and using the following estimators: the sample matrix, one shrinkage estimator and one portfolio of estimators. Again, the ex- post GMVP is used as our betterment criterion.

Like results reported in Bengtsson and Holst (2002) and Jagannathan and Ma (2003), we also find that when the short sale constraints are imposed, all three estimators perform substantially better than the sample matrix when the constraints are not imposed. Not surprisingly, however, imposing short sale constraints has a cost: When compared to the unconstrained GMVP constructed from the shrinkage estimators and the portfolios of estimators, the GMVP has significantly higher variance in the presence of the short sale constraints. This is true no matter which of the three estimators we use when the short sale constraints are imposed. This statistically-significant gap between the performances of the estimators when the short sale constraints are imposed and not imposed is the “price” of not holding short sale positions.[9]

Our findings differ from those of Bengtsson and Holst (2002) and Jagannathan and Ma (2003) in at least one significant respect: When the short sale constraints are imposed, both the shrinkage estimator and the portfolio of estimators perform statistically significantly better than the sample matrix. We also find that, when imposing the constraints, the portfolio of estimators performs at least as well as the more sophisticated shrinkage estimator. This again confirms our notion that simpler is better, at least when it comes to shrinkage.

Imposing the short sale constraints essentially means that every portfolio weight, which would otherwise be negative (no matter how large or small), is set to zero, and that other portfolio weights are computed accordingly. This also means that small changes in the

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[9] Levy and Ritov (2001) also discuss the “price” of not holding short positions. They measure this price by comparing the Sharpe ratios of the optimal portfolios when the short sale constraints are imposed and not imposed.

covariance matrix (and/or the asset expected returns) do not affect the assets held in zero position. The no short sale constraints therefore impose a severe discontinuity on the relation between asset statistics and optimal asset weights. In addition, when the short sale constraints are imposed, the solution obtained for the portfolio weights is numerical and non-analytical.

In order to try and overcome the drawbacks of the short sale constraints, we introduce a new estimator of the covariance matrix, which produces—in an unconstrained optimization—a positive GMVP. The GMVP weights constructed from our new estimator can be found analytically. In addition, they can be sensitive to even small changes in the covariance matrix, so that the "discontinuity" of the short sale constraints can be reduced, when our new estimator is used instead. We call our new estimator the "two-block estimator" and we construct it as follows: We divide the estimated covariance matrix into two blocks, and each block has the sample variances on the diagonal. Pairs of stocks within the same block have the same covariance, and the covariance between stocks from different blocks equals a third constant. We derive analytically sufficient conditions for a matrix of this type to produce positive GMVP without imposing the short sale constraints. These sufficient conditions allow for a rich set of covariance matrices, what makes our new two-block estimator useful.[10]

As we also wish to evaluate the performance of our new two-block estimator, we add to our "horse races" an arbitrary example of the two-block estimator. We find that the arbitrary example of the two-block estimator performs at least as well as a combination of imposing the

[10] The question of non-negative portfolios has been widely recognized as an important problem in portfolio optimization. Previous papers by Roll and Ross (1977), Rudd (1977), Green (1986) and Nielsen (1987) have established conditions that give non-negative frontier portfolios. Our theorem differs from this literature in that it establishes a constructive test for a covariance matrix which gives a positive GMVP.

short sale constraints and using the sample matrix. It performs slightly less well than the portfolio of estimators and the shrinkage estimator when the constraints are imposed, though mostly not statistically significantly less well than the shrinkage estimator. Since our sufficient conditions allow for a rich set of covariance matrices, it is possible to construct a two-block estimator, which is based on a more solid financial and statistical ground than our arbitrary example. We predict that such a two-block estimator will perform even better. As an aside, placing our example of the two-block estimator in a portfolio of estimators, together with the sample matrix and the single-index matrix, produces the estimator that performs the best throughout our whole performance contest, though not statistically significantly better than the other portfolios of estimators and the shrinkage estimators, when the short sale constraints are not imposed.

The remainder of this paper proceeds as follows. In section 2 we present in a nutshell the mathematics of the MV theory, which is the basis of our work. In section 3 we describe our "horse race" between the shrinkage estimators and the portfolios of estimators when the short constraints are not imposed. In section 4 we discuss our "horse race" when the short sale constraints are imposed. In section 5 we introduce our new two-block estimator that produces a positive GMVP without imposing the short sale constraints. Section 6 empirically evaluates the performance of an arbitrary example of the two-block estimator. We conclude the paper with a brief summary in section 7.

2. Mean-variance theory for portfolio selection

We consider a universe of N stocks whose returns are distributed with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.[11] Following Markowitz (1952, 1959) we define the problem of portfolio selection as follows:

$$\begin{aligned}
&\underset{\mathbf{w}}{\text{Min}}\ \mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w}\\
&\text{subject to}\\
&\mathbf{w}^{\mathbf{T}}\boldsymbol{\mu} = q\\
&\mathbf{w}^{\mathbf{T}}\mathbf{1} = 1
\end{aligned}$$

where $\mathbf{w}$ denotes the vector of portfolio weights, q denotes the expected return that is required on the portfolio and $\mathbf{1}$ denotes a vector of ones. Solving the first order conditions by using the Lagrange multipliers leads to the well-known solution:

$$\mathbf{w} = \frac{C - qB}{AC - B^2}\boldsymbol{\Sigma}^{\mathbf{-1}}\mathbf{1} + \frac{qA - B}{AC - B^2}\boldsymbol{\Sigma}^{\mathbf{-1}}\boldsymbol{\mu}$$

where $\boldsymbol{\Sigma}^{\mathbf{-1}}$ denotes the inverse matrix of $\boldsymbol{\Sigma}$ and $A = \mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{\mathbf{-1}}\mathbf{1}$; $B = \mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{\mathbf{-1}}\boldsymbol{\mu}$; $C = \boldsymbol{\mu}^{\mathbf{T}}\boldsymbol{\Sigma}^{\mathbf{-1}}\boldsymbol{\mu}$.

We can see that since the optimal portfolio weights depend on the inverse of the covariance matrix, practical implementation of the MV theory actually requires the usage of an invertible covariance matrix estimator. Except for the global minimum variance portfolio (GMVP), finding the optimal portfolio weights requires the estimation of both the expected stock returns vector and the covariance matrix. In order to focus on the estimation of the covariance matrix, in this paper we look only at the GMVP, which does not require the estimation of the

[11] Throughout this paper, vectors (matrices) are denoted in lower (upper) case boldface. The superscript **T** denotes the transpose of a matrix or a vector.

expected stock returns vector. In order to obtain the GMVP, we omit the constraint $\mathbf{w}^{\mathbf{T}}\boldsymbol{\mu} = q$; therefore, the portfolio selection problem becomes:

$$\begin{aligned} &\underset{\mathbf{w}}{\mathrm{Min}}\ \mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w} \\ &\text{subject to} \\ &\mathbf{w}^{\mathbf{T}}\mathbf{1} = 1 \end{aligned}$$

After solving again the first order conditions by using the Lagrange multipliers, the GMVP is given by:

$$\mathbf{w}_{GMVP} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

If short sale constraints are imposed, the portfolio selection problem becomes:

$$\begin{aligned} &\underset{\mathbf{w}}{\mathrm{Min}}\ \mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\mathbf{w} \\ &\text{subject to} \\ &\mathbf{w}^{\mathbf{T}}\mathbf{1} = 1 \\ &\mathbf{w} \geq 0 \end{aligned}$$

Because the optimization problem in this form includes inequality constraints, it cannot be solved analytically. In order to generate a solution, an iterative procedure, based on the Kuhn-Tucker conditions, is commonly used.

3. The performance contest—part 1

In this section we run a "horse race" between various shrinkage estimators and portfolios of estimators, when the short sale constraints are not imposed. We show empirically that there is no statistically-significant gain from using the more sophisticated shrinkage methods—all of the methods discussed in this section lead to similar improvements in estimating the GMVP. We conclude that simpler is better, at least when it comes to shrinkage.

3.1. Data and period of study

We use monthly returns on stocks traded on the New York Stock Exchange (NYSE). The stock returns are extracted from the Center for Research in Security Prices (CRSP) database. The period of the study is from 1/1964 to 12/2003.

3.2. Methodology

For evaluating the performance of the estimators included in our "horse race," one can compare the estimators computed over a certain sample period (the in-sample period) with the covariance matrix realized over a subsequent period (the out-of-sample period). However, our main interest is to assess how the performance of the estimators translates into the performance of the optimal portfolios obtained from the MV optimization process. Therefore, we find it more useful to conduct an empirical performance contest that focuses on the out-of-sample performance of the respective optimal portfolios. We follow Chan et al. (1999), Bengtsson and Holst (2002), Jagannathan and Ma (2003) and Ledoit and Wolf (2003) and use the ex-post GMVP as our betterment criterion. We benefit from the fact that finding the GMVP does not require the estimation of the expected stock returns vector, which is out of the scope of our paper.[12]

In our performance contest we mimic an investor having the following investment strategy:

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[12] Elton et al. (2005) point out that evaluating the performance of the covariance matrix estimators based on the ex-post GMVP suffers from some bias, since the stocks enter the GMVP are principally the ones with low correlations. Basak et al. (2004) state that the estimate of the variance of a GMVP constructed using an estimated covariance matrix will on average be strictly smaller than its true variance.

1. The investor is only interested in stocks traded on the NYSE.
2. He only cares about minimizing the risk of his investment and therefore invests in the GMVP.
3. He chooses a method to estimate the covariance matrix based on historical data of stock returns. He chooses the length of the backward period, in which he collects monthly return data on stocks traded on the NYSE (the in-sample-period). Based on this data he computes the estimator of the covariance matrix, finds the GMVP and invests in this portfolio.
4. He has a certain investment horizon. Namely, he keeps his portfolio unchanged for a certain period (the out-of-sample period). When this period is over, he liquidates the portfolio. Then, he starts the whole process of estimating the covariance matrix, constructing the GMVP and holding it until liquidation all over again.

To illustrate the way we conduct our performance contest, let us assume that the first time our investor wishes to invest is January 1974, and that he chooses an in-sample period of 120 months and an out-of-sample period of 12 months. Therefore:

1. We collect monthly return data of stocks traded on the NYSE from 1/64 till 12/73.

   We choose an estimator of the covariance matrix, which is computed based on this data.
2. We construct the GMVP from the estimator computed in phase 2.
3. We record the monthly returns on the GMVP From 1/74 till 12/74.
4. We start the whole process all over again. Namely, we collect monthly return data of stocks traded on the NYSE from 1/65 till 12/74, based on this data we

compute the same estimator used in phase 1, construct the GMVP and record its monthly returns from 1/75 till 12/75 and so on…

5. We repeat the process of computing the covariance matrix, constructing the GMVP and recording its monthly returns in the out-of-sample period 30 times (the last monthly return recorded is from 12/2003). As a result, all together, we collect 360 monthly returns (from 1/74 till 12/2003) on the GMVP.
6. We compute the standard deviation of the collected 360 monthly returns. This standard deviation represents the risk our investor was exposed to in the 30 years he was running his investment strategy. Given the chosen in-sample and out-of-sample periods, we can refer to the computed standard deviation as a proxy of the performance of the specific estimator used for estimating the covariance matrix.

We conduct our test for seven different estimators. Since the motivation of our investor is to minimize the risk of his investment, the smaller the standard deviation of the collected 360 monthly returns, the better the respective estimator of the covariance matrix.

We run our "horse race" six times, each time changing the length of the in-sample period or the length of the out-of-sample period. We use in-sample periods of 120 months (also used in Ledoit and Wolf [2003]) and 60 months (also used in Chan et al. [1999] and Jagannathan and Ma [2003]).[13] We use out-of-sample periods of 12 months (also used in Chan et al. [1999], Jagannathan and Ma [2003] and Ledoit and Wolf [2003]), 24 months and 36 months. We chose these three out-of-sample periods, since we believe they correspond to realistic investment horizons (see also Chan et al. [1999]). As an aside, we always construct the first GMVP on 1/74

[13] Jobson and Korkie (1981) mention rules of thumb regarding the length of the in-sample period of 4 to 7 years and 8 to 10 years.

and record the last return data on the last GMVP on 12/03. Thus, in each of the six runs for every one of the seven estimators participating in our contest, we have a set of 360 monthly returns used to compute the respective standard deviation.

It is also worth mentioning that each time we construct a GMVP, we construct it only out of NYSE stocks whose returns cover the entire in-sample and out-of-sample periods used. For example, in the case of in-sample period of 120 months and out-of sample period of 12 months, for constructing the GMVP of 1/74, we only use NYSE stocks with monthly return data for all the 132 months from 1/64 till 12/74. For constructing the GMVP of 1/75, we only use NYSE stocks with monthly return data for all the 132 months from 1/65 till 12/75 and so on. Therefore, the resulting number of stocks used for constructing the GMVP varies across the years and the runs of the contest (see also Bengtsson and Holst [2002]).[14]

3.3. The covariance matrix estimators included in our "horse race"

We focus on the following seven covariance matrix estimators:

Shrinkage to the single-index model: This is the shrinkage estimator suggested by Ledoit and Wolf (2003), in which the covariance matrix estimator obtained from Sharpe's (1963) single-index model (henceforth—the single-index matrix) joins the sample matrix in the

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[14] We are aware of the fact that this widely-followed procedure introduces survivorship bias into the estimation procedure. However, since the survivorship bias is common to all the compared estimators, we do not consider this a significant problem.

weighted average. This estimator performed best in Ledoit and Wolf's (2003) contest and best in Jagannathan and Ma's (2003) contest.[15]

Shrinkage to the constant correlation model: This is the shrinkage estimator suggested by Ledoit and Wolf (2004b), in which the covariance matrix estimator obtained by assuming that each pair of stocks has the same correlation (henceforth—the constant correlation matrix) joins the sample matrix in the weighted average. Ledoit and Wolf (2004b) find the performance of this estimator comparable to the performance of the shrinkage to the single-index model estimator.

A portfolio of the sample matrix, the single-index matrix and the diagonal matrix: This is the estimator, which had been introduced by Jagannathan and Ma (2000) and adopted by Bengtsson and Holst (2002). It consists of an equally weighted average of the sample matrix, the single-index matrix and a matrix containing only the diagonal elements of the sample matrix (henceforth—the diagonal matrix). Its performance was found to be one of the best in Bengtsson and Holst's (2002) contest.

A portfolio of the sample matrix, the single-index matrix and the constant correlation matrix: This estimator has not been previously been used in the literature. It consists of an equally weighted average of the sample matrix, the single-index matrix and the constant correlation matrix.

A portfolio of the sample matrix, the single-index matrix, the constant correlation matrix and the diagonal matrix: This estimator has not been introduced in the literature before. It

[15] In fact, in Jagannathan and Ma's (2003) contest, the shrinkage estimator performed best together with the sample covariance matrix based on daily return data, which is out of the scope of this paper.

consists of an equally weighted average of the sample matrix, the single-index matrix, the constant correlation matrix and the diagonal matrix.

A random average of the sample matrix and the single-index matrix: As in the case of the shrinkage to the single-index model estimator, this estimator is based on a weighted average of the sample matrix and the single-index matrix. However, this time the proportion of the single-index matrix in the weighted average is drawn from a uniform distribution on the interval (0.5,1). We include this estimator in our contest, because it helps us highlight the problem of estimating the proportions of the estimators in the shrinkage estimators (see also Bengtsson and Holst [2002] and Jagannathan and Ma [2003]). Ledoit and Wolf (2003) and Bengtsson and Holst (2002) point out that there is more estimation error in the sample matrix than there is specification error in the single-index matrix. Therefore, we allow the proportion of the single-index matrix in the random average to obtain only values greater than 0.5.

The diagonal matrix: This estimator contains a lot of specification error, and clearly does not obey the statistical principle of reducing the estimation error of the sample matrix without creating too much specification error instead. It is included in the contest as our "stalking horse," since for our data the sample matrix is not invertible and cannot therefore be used.

3.4. Results

In Tables 1 and 2 below we report the out-of-sample standard deviations. The standard deviations are annualized through multiplication by $\sqrt{12}$.

SUMMARY IN SAMPLE 120 MONTHS				
In Sample 120 Months Out Of Sample 12 Months			**In Sample 120 Months Out Of Sample 36 Months**	
Average Size of Universe of Stocks	900.7		Average Size of Universe of Stocks	806
The Largest Universe	1063		The Largest Universe	865
The Smallest Universe	795		The Smallest Universe	729
	Annual σ			Annual σ
Diagonal	13.12%		Diagonal	13.16%
Shrinkage to constant correlation	8.52%		Random average of sample and single index	9.00%
Random average of sample and single index	8.51%		Shrinkage to single index	8.94%
Portfolio of sample, single index, constant correlation	8.47%		Portfolio of sample, single index, diagonal	8.93%
Portfolio of sample, single index, constant correlation, diagonal	8.46%		Shrinkage to constant correlation	8.91%
Portfolio of sample, single index, diagonal	8.39%		Portfolio of sample, single index, constant correlation	8.85%
Shrinkage to single index	8.37%		Portfolio of sample, single index, constant correlation, diagonal	8.81%
Gap between Worst and Best Improver	0.15%		Gap between Worst and Best Improver	0.19%
In Sample 120 Months Out Of Sample 24 Months				
Average Size of Universe of Stocks	853.1			
The Largest Universe	945			
The Smallest Universe	769			
	Annual σ			
Diagonal	13.12%			
Shrinkage to constant correlation	8.97%			
Random average of sample and single index	8.93%			
Portfolio of sample, single index, constant correlation	8.90%			
Portfolio of sample, single index, diagonal	8.89%			
Shrinkage to single index	8.89%			
Portfolio of sample, single index, constant correlation, diagonal	8.85%			
Gap between Worst and Best Improver	0.12%			

Table 1: The annualized out-of-sample standard deviations generated by each of the seven tested covariance matrix estimators in the three runs of the in-sample period of 120 months.

SUMMARY IN SAMPLE 60 MONTHS				
In Sample 60 Months Out Of Sample 12 Months			**In Sample 60 Months Out Of Sample 36 Months**	
Average Size of Universe of Stocks	1260.9		Average Size of Universe of Stocks	1110.1
The Largest Universe	1739		The Largest Universe	1424
The Smallest Universe	1029		The Smallest Universe	958
	Annual σ			Annual σ
Diagonal	12.90%		Diagonal	12.84%
Random average of sample and single index	8.47%		Portfolio of sample, single index, diagonal	8.91%
Shrinkage to constant correlation	8.46%		Shrinkage to single index	8.90%
Portfolio of sample, single index, constant correlation	8.40%		Random average of sample and single index	8.89%
Portfolio of sample, single index, constant correlation, diagonal	8.37%		Shrinkage to constant correlation	8.83%
Portfolio of sample, single index, diagonal	8.34%		Portfolio of sample, single index, constant correlation	8.81%
Shrinkage to single index	8.31%		Portfolio of sample, single index, constant correlation, diagonal	8.78%
Gap between Worst and Best Improver	0.16%		Gap between Worst and Best Improver	0.13%
In Sample 60 Months Out Of Sample 24 Months				
Average Size of Universe of Stocks	1186.1			
The Largest Universe	1572			
The Smallest Universe	985			
	Annual σ			
Diagonal	12.92%			
Portfolio of sample, single index, diagonal	8.94%			
Shrinkage to single index	8.91%			
Shrinkage to constant correlation	8.85%			
Portfolio of sample, single index, constant correlation	8.83%			
Random average of sample and single index	8.83%			
Portfolio of sample, single index, constant correlation, diagonal	8.81%			
Gap between Worst and Best Improver	0.13%			

Table 2: The annualized out-of-sample standard deviations generated by each of the seven tested covariance matrix estimators in the three runs of the in-sample period of 60 months.

The six estimators all perform substantially better than the "stalking horse," the diagonal matrix. The most important point of our horse race, however, is not the improvement of the six methods over the (obviously inferior) diagonal matrix. The most important point is that all of the estimators we examine perform within the same range. In other words, we find very little qualitative difference between any of the estimators: The largest gap in a specific run between the standard deviations corresponding to the "best" and "worst" improvements of our six estimators is obtained in the case of in-sample period of 120 months and out-of-sample period of 36 months and it is only 0.19%. Checking, for each run, whether the tiny differences in the performance of the various estimators are statistically significant results in a negative answer in all cases. In addition, we can see that the ranking of the estimators changes from one run to another, and in fact the portfolio of the sample matrix, the single-index matrix, the constant correlation matrix and the diagonal matrix "wins" the contest four times, whereas the shrinkage to the single-index model estimator "wins" only twice.

We conclude that all six of the estimators have substantially the same performance improvement, and that therefore there is no need to use the methodologically complex shrinkage estimators. Instead, one can simply use the portfolio of estimators, as long as these portfolios obey the statistical principle of reducing the estimation error of the sample matrix without creating too much specification error instead.

One could claim that if a better estimator than the market matrix or the constant correlation matrix is found, then the shrinkage estimator relaying on this estimator will perform better than any portfolio of estimators. We do not agree with such a claim. In our opinion, placing this estimator in a portfolio of estimators will result in a performance within the same range as of the shrinkage estimator. An example for that can be found in the work of Bengtsson and Holst (2002). They develop a rather complicated shrinkage estimator, in which the estimator joining the sample matrix in the weighted average is generated from principal component analysis. According to them, their new shrinkage estimator performs better than the shrinkage estimator of Ledoit and Wolf (2003). However, when they place their developed estimator in a portfolio together with the sample matrix and the diagonal matrix, they obtain an estimator that performs at least as good as their shrinkage estimator.

We can see that both the random average of the sample matrix and the single-index matrix and the shrinkage to the single-index model estimator perform within the same range. Theoretically, the shrinkage estimator should perform better than any other weighted average of the two estimators, since the proportions in the weighted average of the shrinkage estimator are obtained from minimizing the quadratic risk (of error) function of the combined estimator. Yet, it seems that in practice, estimating these specific proportions gives rise to a new type of error, and overall the shrinkage estimator does not perform better than the random average. This result is similar to the one obtained regarding this issue in Jagannathan and Ma (2003). We suspect Bengtsson and Holst (2002) report a contradicted result, because they use for their random average a uniform distribution on the interval (0,1), and not on the interval (0.5,1). Thus, they do not prevent situations, in which the proportion of the single-index matrix is smaller than the proportion of the sample matrix. In these cases, the estimator obtained contains probably too much specification error, and therefore cannot compete with the shrinkage estimator.

As an aside, we can notice that when keeping the length of the out-of-sample period fixed and changing the in-sample period from 120 to 60 months, we obtain quite similar results for the performance of the various estimators. In addition, when keeping the in-sample period fixed and changing the out-of-sample period from 12 months to 24 or 36 months, the annualized standard deviations grow in approximately 0.5%, though not statistically-significant in a significant level of 5%. We believe future research should address more carefully the effects (if any) of the chosen length of the in-sample and out-of-sample periods on the performance of the covariance matrix estimators.

4. The performance contest—part 2

So far we have evaluated the covariance matrix estimators without addressing the issue of short selling positions. In this section we discuss this issue and the effects of imposing the short sale constraints on the portfolio selection problem.

4.1. Average short positions

In Table 3 we present the average amount of short positions obtained for each estimator in all six runs of our contest. The amount of short positions is defined as the sum of all negative portfolio weights.

Average Short Positions						
Covariance Matrix Estimator	*In Sample / Out of Sample*					
	120 / 12	120 / 24	120 / 36	60 / 12	60 / 24	60 / 36
Diagonal	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Random average of sample and single index	-90.09%	-91.40%	-87.84%	-62.01%	-63.37%	-64.21%
Portfolio of sample, single index, constant correlation	-117.17%	-117.46%	-116.50%	-91.36%	-92.10%	-94.46%
Portfolio of sample, single index, diagonal	-90.21%	-90.47%	-89.64%	-65.22%	-65.86%	-67.87%
Portfolio of sample, single index, constant correlation, diagonal	-101.41%	-101.23%	-99.99%	-84.04%	-84.43%	-86.30%
Shrinkage to constant correlation	-129.07%	-130.24%	-130.94%	-92.76%	-93.41%	-96.44%
Shrinkage to single index	-96.96%	-97.96%	-97.84%	-66.55%	-67.19%	-69.29%

Table 3: Average short positions—total of all negative portfolio proportions—generated by each of our seven tested covariance matrix estimators. An average short interest of -96.96%, obtained for the shrinkage to the single-index model estimator in the run of in-sample period of 120 months and out-of-sample period of 12 months, means that in average, over the 30 portfolios constructed in this run based on this estimator, for every dollar invested in the portfolio we short 96.96 cents worth of stocks, while buying $1.9696 worth of other stocks.

We can see that, apart from the estimator based on the diagonal matrix, which generates a positive GMVP, all other estimators in all the runs of the contest generate portfolios with significant short sale positions. Moving from in-sample period of 120 months to in-sample period of 60 months reduces the average short positions for each one of the estimators; however even then we are still talking about quite significant short sale positions. Also Bengtsson and Holst (2002), Jagannathan and Ma (2003) and Ledoit and Wolf (2003) report significant short positions in their performance contests.

To the extent that short sales are considered an undesirable feature of portfolio optimization, the shrinkage estimators and the portfolios of estimators cannot satisfy us anymore. We cannot count on the diagonal estimator either, since it generates relatively high out-of sample standard deviations (see Tables 1 and 2). Hence, our goal now is to check whether we can find an estimation method that generates both low out-of-sample standard deviations and positive portfolios. One candidate for such a method is the addition of the short sale constraints to the

GMVP problem. In the next subsection we examine empirically the attractiveness of this method.

4.2. Imposing the short sale constraints

We again run our "horse race," this time imposing the short sale constraints. We use three covariance matrix estimators—the sample matrix, the shrinkage to the single-index model estimator and the portfolio of the sample matrix, the single-index matrix, the constant correlation matrix and the diagonal matrix.[16] As before, our "stalking horse" is the diagonal matrix and the ex- post GMVP is used as our betterment criterion. In Tables 4 and 5 below we report the out-of-sample standard deviations. The standard deviations are annualized through multiplication by $\sqrt{12}$.

[16] In order to find the GMVP weights, we use the Mosek iterative procedure together with the Matlab program. When the diagonal matrix is used, the short sale constraints are of course not needed, as the diagonal matrix anyhow generates a positive GMVP.

Summary In Sample 120 Months When Short Sale Constraints are Imposed				
In Sample 120 Months Out Of Sample 12 Months			**In Sample 120 Months Out Of Sample 36 Months**	
Average Size of Universe of Stocks	900.7		Average Size of Universe of Stocks	806
The Largest Universe	1063		The Largest Universe	865
The Smallest Universe	795		The Smallest Universe	729
	Annual σ			Annual σ
Diagonal	13.12%		Diagonal	13.16%
Sample	10.74%		Sample	11.08%
Shrinkage to single index	9.94%		Shrinkage to single index	10.23%
Portfolio of sample, single index, constant correlation, diagonal	9.65%		Portfolio of sample, single index, constant correlation, diagonal	10.06%
In Sample 120 Months Out Of Sample 24 Months				
Average Size of Universe of Stocks	853.1			
The Largest Universe	945			
The Smallest Universe	769			
	Annual σ			
Diagonal	13.12%			
Sample	10.87%			
Shrinkage to single index	10.19%			
Portfolio of sample, single index, constant correlation, diagonal	10.01%			

Table 4: The annualized out-of-sample standard deviations generated by each of the four tested covariance matrix estimators in the three runs of the in-sample period of 120 months when the short sale constraints are imposed.

Summary In Sample 60 Months When Short Sale Constraints are Imposed				
In Sample 60 Months Out Of Sample 12 Months			**In Sample 60 Months Out Of Sample 36 Months**	
Average Size of Universe of Stocks	1260.9		Average Size of Universe of Stocks	1110.1
The Largest Universe	1739		The Largest Universe	1424
The Smallest Universe	1029		The Smallest Universe	958
	Annual σ			Annual σ
Diagonal	12.90%		Diagonal	12.84%
Sample	10.84%		Sample	11.48%
Shrinkage to single index	9.75%		Shrinkage to single index	10.17%
Portfolio of sample, single index, constant correlation, diagonal	9.15%		Portfolio of sample, single index, constant correlation, diagonal	9.57%
In Sample 60 Months Out Of Sample 24 Months				
Average Size of Universe of Stocks	1186.1			
The Largest Universe	1572			
The Smallest Universe	985			
	Annual σ			
Diagonal	12.92%			
Sample	11.23%			
Shrinkage to single index	10.24%			
Portfolio of sample, single index, constant correlation, diagonal	9.65%			

Table 5: The annualized out-of-sample standard deviations generated by each of the four tested covariance matrix estimators in the three runs of the in-sample period of 60 months when the short sale constraints are imposed.

As in the studies of Bengtsson and Holst (2002) and Jagannathan and Ma (2003), our results confirm that imposing the short sale constraints substantially reduces the out-of-sample standard deviations compared to the GMVP of our "stalking horse," the diagonal matrix. Not surprisingly, however, imposing short sale constraints has a cost, which is revealed when one compares the out-of-sample standard deviations generated by the shrinkage estimator and the portfolio of estimators when the short sale constraints are imposed and when they are not imposed. In the case of the shrinkage estimator, imposing the constraints increases the out-of-sample standard deviations by about 1.3% to 1.6%. In the case of the portfolio of estimators, imposing the constraints increases the out-of-sample standard deviations by about 1.2% when in-sample periods of 120 months are used and by about 0.8% when in-sample periods of 60 months are used. All gaps in all runs for both estimators are statistically-significant. This statistically-significant gap between the performances of the estimators when the short sale constraints are imposed and not imposed is the "price" of not holding short sale positions.

Our findings differ from those of Bengtsson and Holst (2002) and Jagannathan and Ma (2003) in at least one significant respect: When the short sale constraints are imposed, both the shrinkage estimator and the portfolio of estimators perform statistically significantly better than the sample matrix.[17] In contrast, applying the same statistical significance tests to the results and samples of Bengtsson and Holst (2002) and Jagannathan and Ma (2003) reveals that in these studies the gaps obtained are not statistically-significant. We believe future research should address this issue more carefully. We can also see that systematically the portfolio of estimators

[17] We use a chi square test and a significance level of 5%. In fact, if we wish to be more precise, in the run of in-sample period of 120 months and out-of-sample period of 24 months, the gap between the shrinkage estimator and the sample matrix lays a little bit outside the border of 5%.

generates lower out-of-sample standard deviations than the shrinkage estimator, though the gaps are not statistically-significant. This again confirms our notion that simpler is better, at least when it comes to shrinkage.

As an aside, it can be noticed that when keeping the out-of-sample period unchanged and reducing the in-sample period from 120 months to 60 months, the portfolio of estimators performs better, the performance of the shrinkage to the single-index model estimator is almost unchanged and the sample matrix deteriorates. All gaps are not statistically-significant in a significant level of 5%. In addition, keeping the in-sample period unchanged and increasing the out-of-sample period from 12 months to 24 or 36 months damage the performance of the three estimators. Again, the gaps are not statistically-significant in a significant level of 5%.

5. The two-block estimator

As explained in Section 1, using the short sale constraints might be considered somewhat less than satisfying—although the GMVP problem $Min\ \mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\,\mathbf{w},\ where\ \mathbf{w} \geq 0\ and\ \mathbf{w}^{\mathbf{T}}\mathbf{1} = 1$ will always have a non-negative solution, this solution is insensitive to small changes in $\boldsymbol{\Sigma}$, and it is a numerical and non-analytical solution. A more satisfying procedure would develop conditions on the covariance matrix, which would produce analytically—in an unconstrained optimization $Min\ \mathbf{w}^{\mathbf{T}}\boldsymbol{\Sigma}\,\mathbf{w}$ —a non-negative GMVP that can be sensitive to even small changes in $\boldsymbol{\Sigma}$. In this section we discuss such a procedure. We present a theorem, which gives sufficient conditions on a new covariance matrix estimator such that the generated unconstrained GMVP has strictly positive weights. We call our new estimator the "two-block estimator" and we construct it as follows: We divide the estimated covariance matrix into two blocks, and each block has the sample variances on the diagonal. Pairs of stocks within the same block have the same covariance, and the covariance between stocks from different blocks equals a third constant.

We define a two-block estimator to be a matrix $\mathbf{\Omega}$ having the following form:

$$\mathbf{\Omega} = \begin{pmatrix} \boxed{\begin{matrix} s_1^2 & \eta_1 & \cdots & \eta_1 \\ \eta_1 & s_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ & & & \eta_1 \\ \eta_1 & \cdots & \eta_1 & s_j^2 \end{matrix}} & \boxed{\begin{matrix} \eta & \cdots & \eta \\ \vdots & \ddots & \vdots \\ \eta & \cdots & \eta \end{matrix}} \\ \boxed{\begin{matrix} \eta & \cdots & \eta \\ \vdots & \ddots & \vdots \\ \eta & \cdots & \eta \end{matrix}} & \boxed{\begin{matrix} s_{j+1}^2 & \eta_2 & \cdots & \eta_2 \\ \eta_2 & s_{j+2}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ & & & \eta_2 \\ \eta_2 & \cdots & \eta_2 & s_n^2 \end{matrix}} \end{pmatrix}$$

Here j is the size of the first block, s_i^2 are the sample variances, η_1 and η_2 are the covariances in block 1 and 2 respectively, and η is the covariance between stocks from different blocks.

The Theorem below characterizes conditions under which the two-block estimator produces unconstrained positive GMVP.

Theorem: Suppose that $\mathbf{\Omega}$ is a two-block estimator for which no two sample variances s_i^2 are equal. Then $\mathbf{\Omega}$ produces a strictly positive GMVP if the following conditions on η_1, η_2 and η hold:

$$\begin{cases} -\left|\eta_1^*\right| < \eta_1 < \min\left(s_i^2\right) & , \quad \left|\eta_1^*\right| < \min\left(s_i^2\right) \;,\; i = 1,\ldots,j \\ -\min\left(s_i^2\right) \le \eta_1 < \min\left(s_i^2\right) & , \quad \left|\eta_1^*\right| \ge \min\left(s_i^2\right) \;,\; i = 1,\ldots,j \end{cases}$$

$$\begin{cases} -\left|\eta_2^*\right| < \eta_2 < \min\left(s_i^2\right) & , \quad \left|\eta_2^*\right| < \min\left(s_i^2\right) \;,\; i = j+1,\ldots,n \\ -\min\left(s_i^2\right) \le \eta_2 < \min\left(s_i^2\right) & , \quad \left|\eta_2^*\right| \ge \min\left(s_i^2\right) \;,\; i = j+1,\ldots,n \end{cases}$$

$$\begin{cases} -\left|\eta_{12}^*\right| < \eta < \left|\eta_{12}^*\right| & , \quad \left|\eta_{12}^*\right| < \min\left(\eta_1,\eta_2\right) \\ -\min\limits_{i=1,\ldots,j}\left(s_i\right)\cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \le \eta \le \min\left(\eta_1,\eta_2\right) & , \quad \left|\eta_{12}^*\right| > \min\limits_{i=1,\ldots,j}\left(s_i\right)\cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \\ -\left|\eta_{12}^*\right| < \eta \le \min\left(\eta_1,\eta_2\right) & , \quad \min\left(\eta_1,\eta_2\right) < \left|\eta_{12}^*\right| \le \min\limits_{i=1,\ldots,j}\left(s_i\right)\cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \end{cases}$$

where $\left|\eta_1^*\right|$ and $\left|\eta_2^*\right|$ are respectively the unique solutions of the following equations:

$$\left|\eta_1^*\right| = 1 \Big/ \sum_{i=1}^{j} \frac{1}{s_i^2 + \left|\eta_1^*\right|} \quad \text{and} \quad \left|\eta_2^*\right| = 1 \Big/ \sum_{i=j+1}^{n} \frac{1}{s_i^2 + \left|\eta_2^*\right|}$$

and:

$$\left|\eta_{12}^*\right| = +\sqrt{\left(\eta_1 + \frac{1}{\sum\limits_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\sum\limits_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}\right)}$$

The theorem is proved in the appendix to the paper. As an aside, the theorem's conditions guarantee not only a positive GMVP but also that the two-block estimator $\mathbf{\Omega}$ has the properties of a true covariance matrix: correlations not greater than |1| and the matrix is positive definite.

If we restrict η_1 and η_2 to be nonnegative, we get even simpler sufficient conditions for which the GMVP is strictly positive:

Corollary: If the covariances η_1 and η_2 in Ω are nonnegative, then sufficient conditions for the GMVP to be strictly positive are given by:

$$0 \le \eta_1 < \min\left(s_i^2\right) \quad , \quad i = 1, \ldots, j$$
$$0 \le \eta_2 < \min\left(s_i^2\right) \quad , \quad i = j+1, \ldots, n$$

$$\begin{cases} -\min\limits_{i=1,\ldots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \le \eta \le \min\left(\eta_1, \eta_2\right) & , \quad \left|\eta_{12}^*\right| > \min\limits_{i=1,\ldots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \\ -\left|\eta_{12}^*\right| < \eta \le \min\left(\eta_1, \eta_2\right) & , \quad \min\left(\eta_1, \eta_2\right) < \left|\eta_{12}^*\right| \le \min\limits_{i=1,\ldots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \end{cases}$$

where: $$\left|\eta_{12}^*\right| = +\sqrt{\left(\eta_1 + \frac{1}{\sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}\right)}$$

The proof of the corollary is straightforward and also brought in the appendix to the paper.

We can see that the sufficient conditions in both the theorem and corollary allow for a wide range of values for η, η_1, and η_2. This makes the two-block estimator useful, as it allows for a rich set of covariance matrices.

The weight of stock *i* in the GMVP, which is derived analytically in the appendix to the paper, is as follows:

$$w_i = \frac{1}{s_i^2 - \eta_1} \cdot \frac{1 + (\eta_2 - \eta) \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} + \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2} + (\eta_1 + \eta_2 - 2\eta) \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}, \; i = 1, \ldots, j$$

$$w_i = \frac{1}{s_i^2 - \eta_2} \cdot \frac{1 + (\eta_1 - \eta) \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}{\sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} + \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2} + (\eta_1 + \eta_2 - 2\eta) \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}, \; i = j+1, \ldots, n$$

We can see that it depends on the sample variances. Hence, the GMVP weights can be sensitive to even small changes in the asset statistics, so that the "discontinuity" of the short sale constraints can be reduced when the two- block estimator is used instead.

6. Empirical evaluation of the two-block estimator

In this section we add to our performance contest an example of the two-block estimator and check whether it produces low out-of-sample standard deviations. We choose arbitrarily looking at the following two-block estimator: Each block includes the same number of stocks, which are divided into the two blocks based on their "permno numbers."[18] The first block consists of stocks with the smaller permno numbers, whereas the second group consists of the stocks with the higher permno numbers. Each time we construct a GMVP, η_1, η_2 and η obtain the following values:

[18] When N is odd, the first block includes one more stock. The permno numbers are attached to every stock reported in the CRSP database, and therefore they are convenient for use.

$$\eta_1 = 0.99 \times \min\left(s_i^2\right) \quad , \quad i = 1, ..., j$$
$$\eta_2 = 0.99 \times \min\left(s_i^2\right) \quad , \quad i = j+1, ..., n$$
$$\eta = 0.99 \times \min\left(\eta_1, \eta_2\right)$$

The results in Table 6 show that our arbitrary example of the two-block estimator performs at least as well as a combination of imposing the short sale constraints and using the sample matrix. In fact, when the in-sample period of 60 months is used the two-block estimator performs statistically significantly better. When the in-sample period of 120 months is used both estimators perform within the same range. The two-block estimator performs slightly less well than the portfolio of estimators and the shrinkage estimator when the no short sale constraints are imposed, though mostly not statistically significantly less well than the shrinkage estimator.

We believe the fact that our sufficient conditions allow for a rich set of covariance matrices leaves enough space for improvement. For example, instead of forming an arbitrary two-block estimator, we can think of a much more financial oriented two-block estimator. In such an estimator, the stocks will be divided into the two (not necessarily equal) blocks based on financial characteristics, such as the beta—the stocks with positive betas will be placed in one block and the stocks with negative betas in the other. Then, η_1 and η_2 would obtain positive values, whereas η would obtain a negative value.[19] We predict that such a two-block estimator will perform better than the arbitrary example presented here. Another direction for improvement could be exploring whether using more than two blocks is beneficial.

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[19] Note that taking positive η_1 and η_2 also enables the usage of the simpler set of sufficient conditions given in the corollary presented in Section 5.

Placing our arbitrary example of the two-block estimator in a portfolio of estimators, together with the sample matrix and the single-index matrix, produces the estimator that performs the best throughout our whole performance contest, though not statistically significantly better than the other portfolios of estimators and the shrinkage estimators, when the short sale constraints are not imposed. When the constraints are imposed, the portfolio including our arbitrary two-block estimator performs within the same range as the shrinkage estimator and the other portfolio of estimators, and statistically significantly better than the sample matrix.

Summary The Two Block Estimator						
Covariance Matrix Estimator	*In Sample / Out of Sample*					
	120 / 12	120 / 24	120 / 36	60 / 12	60 / 24	60 / 36
Two block estimator	10.59%	11.19%	10.97%	9.98%	10.46%	10.50%
Portfolio of sample, single index, two block	8.08%	8.67%	8.65%	8.02%	8.76%	8.71%
Portfolio of sample, single index, two block, constraints imposed	9.75%	10.05%	10.09%	9.45%	10.10%	10.08%

Table 6: The annualized out-of-sample standard deviations generated by the two-block estimator when it is used solely, when it is added to a portfolio of estimators and when it is added to a portfolio of estimators and the no short sale constraints are imposed.

We do not report here results from another estimator that produces positive efficient portfolios without imposing the short sale constraints.[20] This estimator consists of a constant correlation coefficient between each pair of stocks and the sample variances. As before, we derived analytically sufficient conditions on the values that the constant correlation coefficient can obtain (given the sample variances), which produce positive GMVP without imposing the short sale constraints. This estimator consistently performed worse than the two-block estimator. This is probably due to the fact that asking for a constant correlation coefficient that generates positive GMVP results in imposing too much structure on the real covariance matrix (giving rise to a significant specification error).

[20] Both the empirical results and the analytics are available from the authors on request.

7. Summary and conclusions

This paper deals with estimating the covariance matrix of stock returns, which is one of the two main elements of the mean-variance theory of Markowitz (1952,1959) for portfolio selection.

Estimating the covariance matrix based solely on the sample matrix is famously difficult, since very often the sample matrix suffers from the "curse of dimensions." As a result, a significant finance literature, which looks for better methods to estimate the covariance matrix, has been spawned. Out of this rich literature, we have chosen to focus, in this paper, on estimators that comply with the following three assumptions: 1. Stock returns are independent and identically distributed (iid). 2. Historical monthly return data should be used in the estimation process. 3. Sample variances are good estimators of the stock variances. Combining the findings of recent studies revels that the best estimators of that type are the shrinkage estimators and the portfolios of estimators.

In our study, we run a "horse race" between various shrinkage estimators and portfolios of estimators. We use the ex-post global minimum variance portfolio (GMVP) as our betterment criterion. We show empirically that all the estimators perform within the same range, and that it is actually impossible to claim that one of them is the better than the other. Hence, there is no statistically-significant gain from using the more sophisticated shrinkage methods, and therefore one can instead use the simpler portfolios of estimators. We conclude that simpler is better, at least when it comes to shrinkage.

A significant drawback of the shrinkage estimators and the portfolios of estimators is that they generate minimum variance portfolios incorporating significant short sale positions. To the extent that short sales are considered an undesirable feature of portfolio optimization, the most intuitive way to overcome them is to add to the portfolio selection problem short sale constraints that prevent the portfolio weights from being negative, no matter which covariance matrix estimator is used.

In our study, we also run a "horse race" in which the short sale constraints are imposed and again the GMVP is used as the betterment criterion. Our findings regarding the short sale constraints differ from previous studies in at least one significant respect: In our sample when the short sale constraints are imposed, both the shrinkage estimator and the portfolio of estimators perform statistically significantly better than the sample matrix. We believe future research should address this issue more carefully. We also find that, when imposing the constraints, the portfolio of estimators performs at least as well as the more sophisticated shrinkage estimator. This again confirms our notion that simpler is better, at least when it comes to shrinkage.

One might find the usage of the short sale constraints somewhat less than satisfying. That is because, when the short sale constraints are imposed, the solution obtained for the GMVP weights is insensitive to small changes in the covariance matrix, as they do not affect the assets held in zero position, and it is a numerical and non-analytical solution. As a result, in this paper, we also introduce a new estimator of the covariance matrix, which produces—in an unconstrained optimization—a positive GMVP. The GMVP weights constructed from our new estimator can be found analytically, and they can be sensitive to even small changes in the covariance matrix.

Our new estimator—the two-block estimator—is constructed as follows: We divide the estimated covariance matrix into two blocks, and each block has the sample variances on the diagonal. Pairs of stocks within the same block have the same covariance, and the covariance between stocks from different blocks equals a third constant. We derive analytically sufficient conditions for a matrix of this type to produce a positive GMVP without imposing the short sale constraints. These sufficient conditions allow for a rich set of covariance matrices, what makes our new two-block estimator useful.

We also make the first step for evaluating the performance of our new two-block estimator. We add to our "horse races" an arbitrary example of the two-block estimator. We find that the arbitrary example of the two-block estimator performs at least as well as a combination of imposing the no short sale constraints and using the sample matrix. It performs slightly less well than the portfolio of estimators and the shrinkage estimator when the constraints are imposed, though mostly not statistically significantly less well than the shrinkage estimator. Since the sufficient conditions in the two-block estimator theorem allow for a rich set of covariance matrices, it is possible to construct a two-block estimator, which is based on a more solid financial and statistical ground than the arbitrary example used here. We predict that such a two-block estimator will perform even better than our arbitrary example. Another direction for improvement could be exploring whether using more than two blocks is beneficial.

As an aside, placing our example of the two-block estimator in a portfolio of estimators, together with the sample matrix and the single-index matrix, produces the best performing "horse" in our whole performance contest, though not statistically significantly better than the other portfolios of estimators and the shrinkage estimators, when the short sale constraints are not imposed.

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Appendix—the Proof of the Theorem and the Corollary

We present here the proof for the theorem and the corollary brought in Section 5.

Theorem: Suppose that $\boldsymbol{\Omega}$ is a two-block estimator for which no two sample variances s_i^2 are equal. Then $\boldsymbol{\Omega}$ produces a strictly positive GMVP if the following conditions on η_1, η_2 and η hold:

$$\begin{cases} -\left|\eta_1^*\right| < \eta_1 < \min\left(s_i^2\right) & , \quad \left|\eta_1^*\right| < \min\left(s_i^2\right) \ , \ i=1,\dots,j \\ -\min\left(s_i^2\right) \le \eta_1 < \min\left(s_i^2\right) & , \quad \left|\eta_1^*\right| \ge \min\left(s_i^2\right) \ , \ i=1,\dots,j \end{cases}$$

$$\begin{cases} -\left|\eta_2^*\right| < \eta_2 < \min\left(s_i^2\right) & , \quad \left|\eta_2^*\right| < \min\left(s_i^2\right) \ , \ i=j+1,\dots,n \\ -\min\left(s_i^2\right) \le \eta_2 < \min\left(s_i^2\right) & , \quad \left|\eta_2^*\right| \ge \min\left(s_i^2\right) \ , \ i=j+1,\dots,n \end{cases}$$

$$\begin{cases} -\left|\eta_{12}^*\right| < \eta < \left|\eta_{12}^*\right| & , \quad \left|\eta_{12}^*\right| < \min\left(\eta_1,\eta_2\right) \\ -\min\limits_{i=1,\dots,j}\left(s_i\right)\cdot\min\limits_{i=j+1,\dots,n}\left(s_i\right) \le \eta \le \min\left(\eta_1,\eta_2\right) & , \quad \left|\eta_{12}^*\right| > \min\limits_{i=1,\dots,j}\left(s_i\right)\cdot\min\limits_{i=j+1,\dots,n}\left(s_i\right) \\ -\left|\eta_{12}^*\right| < \eta \le \min\left(\eta_1,\eta_2\right) & , \quad \min\left(\eta_1,\eta_2\right) < \left|\eta_{12}^*\right| \le \min\limits_{i=1,\dots,j}\left(s_i\right)\cdot\min\limits_{i=j+1,\dots,n}\left(s_i\right) \end{cases}$$

where $\left|\eta_1^*\right|$ and $\left|\eta_2^*\right|$ are respectively the unique solutions of the following equations:

$$\left|\eta_1^*\right| = 1\Big/\sum_{i=1}^{j}\frac{1}{s_i^2+\left|\eta_1^*\right|} \quad \text{and} \quad \left|\eta_2^*\right| = 1\Big/\sum_{i=j+1}^{n}\frac{1}{s_i^2+\left|\eta_2^*\right|}$$

and:

$$\left|\eta_{12}^*\right| = +\sqrt{\left(\eta_1+\frac{1}{\sum\limits_{i=1}^{j}\frac{1}{s_i^2-\eta_1}}\right)\left(\eta_2+\frac{1}{\sum\limits_{i=j+1}^{n}\frac{1}{s_i^2-\eta_2}}\right)}$$

Corollary: If the covariances η_1 and η_2 in Ω are nonnegative, then sufficient conditions for the GMVP to be strictly positive are given by:

$$0 \le \eta_1 < \min\left(s_i^2\right) \quad , \quad i = 1, \ldots, j$$

$$0 \le \eta_2 < \min\left(s_i^2\right) \quad , \quad i = j+1, \ldots, n$$

$$\begin{cases} -\min\limits_{i=1,\ldots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \le \eta \le \min\left(\eta_1, \eta_2\right) & , \quad \left|\eta_{12}^*\right| > \min\limits_{i=1,\ldots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \\ -\left|\eta_{12}^*\right| < \eta \le \min\left(\eta_1, \eta_2\right) & , \quad \min\left(\eta_1, \eta_2\right) < \left|\eta_{12}^*\right| \le \min\limits_{i=1,\ldots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \end{cases}$$

where: $$\left|\eta_{12}^*\right| = +\sqrt{\left(\eta_1 + \frac{1}{\sum\limits_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\sum\limits_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}\right)}$$

Proof:

The proof consists of three phases:

1. Finding sufficient conditions on η_1, η_2 and η for which a strictly positive GMVP is obtained.
2. Modifying the conditions, in order to guarantee that the obtained correlation coefficient between any pair of stocks is not greater than |1|.
3. Modifying the conditions once again, in order to guarantee that the matrix is positive definite (i.e. its eigenvalues are strictly positive).

The last two phases are conducted, in order to guarantee that the two-block estimator has the properties of a true covariance matrix.

Phase 1—strictly positive GMVP

Denote the first column of $\mathbf{\Omega}^{-1}$ by: $\begin{pmatrix} x_1 \\ \vdots \\ \\ x_j \\ x_{j+1} \\ \vdots \\ \\ \\ x_n \end{pmatrix}$

where $\mathbf{\Omega}^{-1}$ denotes the inverse matrix of $\mathbf{\Omega}$.

Then, since $\mathbf{\Omega}\mathbf{\Omega}^{-1} = \mathbf{I}$:

$$\mathbf{\Omega} \cdot \begin{pmatrix} x_1 \\ \vdots \\ \\ x_j \\ x_{j+1} \\ \vdots \\ \\ \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \\ \\ \\ \\ \\ 0 \end{pmatrix}$$

Given the structure of $\mathbf{\Omega}$, writing this explicitly yields:

$$\text{(1)} \qquad \begin{aligned}
&s_1^2 x_1 + \eta_1 x_2 + \ldots + \eta_1 x_j + \eta x_{j+1} + \eta x_{j+2} \ldots + \eta x_n = 1 \\
&\eta_1 x_1 + s_2^2 x_2 + \ldots + \eta_1 x_j + \eta x_{j+1} + \eta x_{j+2} \ldots + \eta x_n = 0 \\
&\vdots \\
&\eta_1 x_1 + \eta_1 x_2 + \ldots + s_j^2 x_j + \eta x_{j+1} + \eta x_{j+2} \ldots + \eta x_n = 0 \\
&\eta x_1 + \eta x_2 + \ldots + \eta x_j + s_{j+1}^2 x_{j+1} + \eta_2 x_{j+2} \ldots + \eta_2 x_n = 0 \\
&\eta x_1 + \eta x_2 + \ldots + \eta x_j + \eta_2 x_{j+1} + s_{j+2}^2 x_{j+2} \ldots + \eta_2 x_n = 0 \\
&\vdots \\
&\eta x_1 + \eta x_2 + \ldots + \eta x_j + \eta_2 x_{j+1} + \eta_2 x_{j+2} \ldots + s_n^2 x_n = 0
\end{aligned}$$

And therefore:

$$
\begin{aligned}
&\left(s_1^2-\eta_1\right)x_1+\eta_1\sum_{i=1}^{j}x_i+\eta\sum_{i=j+1}^{n}x_i=1\\
&\left(s_2^2-\eta_1\right)x_2+\eta_1\sum_{i=1}^{j}x_i+\eta\sum_{i=j+1}^{n}x_i=0\\
&\vdots\\
&\left(s_j^2-\eta_1\right)x_j+\eta_1\sum_{i=1}^{j}x_i+\eta\sum_{i=j+1}^{n}x_i=0\\
&\eta\sum_{i=1}^{j}x_i+\left(s_{j+1}^2-\eta_2\right)x_{j+1}+\eta_2\sum_{i=j+1}^{n}x_i=0\\
&\eta\sum_{i=1}^{j}x_i+\left(s_{j+2}^2-\eta_2\right)x_{j+2}+\eta_2\sum_{i=j+1}^{n}x_i=0\\
&\vdots\\
&\eta\sum_{i=1}^{j}x_i+\left(s_n^2-\eta_2\right)x_n+\eta_2\sum_{i=j+1}^{n}x_i=0
\end{aligned}
$$

Dividing the first *j* equations by $s_i^2-\eta_1$ (assuming $\eta_1 \neq s_i^2$, $i=1,...,j$) and dividing the last *n*-*j* equations by $s_i^2-\eta_2$ (assuming $\eta_2 \neq s_i^2$, $i=j+1,...,n$) give:

$$
(2)\qquad
\begin{aligned}
&x_1+\frac{\eta_1}{s_1^2-\eta_1}\sum_{i=1}^{j}x_i+\frac{\eta}{s_1^2-\eta_1}\sum_{i=j+1}^{n}x_i=\frac{1}{s_1^2-\eta_1}\\
&\vdots\\
&x_j+\frac{\eta_1}{s_j^2-\eta_1}\sum_{i=1}^{j}x_i+\frac{\eta}{s_j^2-\eta_1}\sum_{i=j+1}^{n}x_i=0\\
&x_{j+1}+\frac{\eta}{s_{j+1}^2-\eta_2}\sum_{i=1}^{j}x_i+\frac{\eta_2}{s_{j+1}^2-\eta_2}\sum_{i=j+1}^{n}x_i=0\\
&\vdots\\
&x_n+\frac{\eta}{s_n^2-\eta_2}\sum_{i=1}^{j}x_i+\frac{\eta_2}{s_n^2-\eta_2}\sum_{i=j+1}^{n}x_i=0
\end{aligned}
$$

Summing the first *j* equations in (2) and rearranging terms give:

$$
(3)\qquad \left(1+\eta_1\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}\right)\sum_{i=1}^{j}x_i+\eta\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}\sum_{i=j+1}^{n}x_i=\frac{1}{s_1^2-\eta_1}
$$

Summing the last *n-j* equations in (2) and rearranging terms give:

$$(4) \qquad \sum_{i=j+1}^{n} x_i = -\frac{\eta \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}} \sum_{i=1}^{j} x_i$$

And now substituting (4) into (3) and rearranging terms give:

$$\sum_{i=1}^{j} x_i = \frac{1}{s_1^2 - \eta_1} \cdot \frac{1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\left(1 + \eta_1 \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}\right)\left(1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}\right) - \eta^2 \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}$$

Denote:

$$\Delta = \left(1 + \eta_1 \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}\right)\left(1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}\right) - \eta^2 \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}$$

And therefore:

$$(5) \qquad \sum_{i=1}^{j} x_i = \frac{1}{s_1^2 - \eta_1} \cdot \frac{1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}$$

Substituting (5) into (4) gives:

$$(6) \qquad \sum_{i=j+1}^{n} x_i = \frac{1}{s_1^2 - \eta_1} \cdot \frac{-\eta \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}$$

Adding equations (5) and (6) gives the sum of (the elements in) the first column of $\mathbf{\Omega}^{-1}$. Since $\mathbf{\Omega}^{-1}$ is symmetric (because $\mathbf{\Omega}$ is symmetric) this is also the sum of (the elements in) the first row of $\mathbf{\Omega}^{-1}$:

$$(7) \qquad \sum_{j=1}^{n} \mathbf{\Omega}_{1j}^{-1} = \frac{1}{s_1^2 - \eta_1} \cdot \frac{1 + (\eta_2 - \eta) \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}$$

Generalizing our last result and finding the sum of each one of the rows of $\mathbf{\Omega}^{-1}$ is done as follows: First, let us look at the first *j* rows of $\mathbf{\Omega}^{-1}$. In order to find the sum of the second row of $\mathbf{\Omega}^{-1}$, we repeat the above procedure. Note that the only difference is that now in (1) the second row and not the first row equals 1 (and of course, the first row now equals 0). In order to find the sum of the third row of $\mathbf{\Omega}^{-1}$, we repeat once again the above procedure. This time the third row in (1) equals 1 and so on. Hence, we repeat the above procedure j times, and each time the only difference is that the row in (1) which equals 1 moves one place ahead. Therefore, the general expression for the sum of one of the first *j* rows of $\mathbf{\Omega}^{-1}$ (let us say row *i*) is:

$$(8) \qquad \sum_{j=1}^{n} \mathbf{\Omega}_{ij}^{-1} = \frac{1}{s_i^2 - \eta_1} \cdot \frac{1 + (\eta_2 - \eta) \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}$$

Repeating the above procedure and applying "symmetric considerations" enable to find the general expression for the sum of one of the last *n-j* rows of $\mathbf{\Omega}^{-1}$ (let us say row *i*):

$$(9) \qquad \sum_{j=1}^{n} \mathbf{\Omega}_{ij}^{-1} = \frac{1}{s_i^2 - \eta_2} \cdot \frac{1 + (\eta_1 - \eta) \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}{\Delta}$$

Recall that the vector of portfolio weights of the GMVP is: $\mathbf{w} = \frac{\mathbf{\Omega}^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\mathbf{\Omega}^{-1}\mathbf{1}}$. For the weight of stock *i*, in the numerator we have the sum of row *i* in $\mathbf{\Omega}^{-1}$ and in the denominator we have the sum of (all elements in) $\mathbf{\Omega}^{-1}$. Hence, in order to obtain the denominator, we use (8) and (9):

$$\mathbf{1}^{\mathrm{T}}\mathbf{\Omega}^{-1}\mathbf{1} = \sum_{i=1}^{j} \left(\frac{1}{s_i^2 - \eta_1} \cdot \frac{1 + (\eta_2 - \eta) \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta} \right) + \sum_{i=j+1}^{n} \left(\frac{1}{s_i^2 - \eta_2} \cdot \frac{1 + (\eta_1 - \eta) \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}{\Delta} \right)$$

And after rearranging a bit more we get:

$$(10) \qquad \mathbf{1}^{\mathrm{T}}\mathbf{\Omega}^{-1}\mathbf{1} = \frac{\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}+\sum_{i=j+1}^{n}\frac{1}{s_i^2-\eta_2}+\left(\eta_1+\eta_2-2\eta\right)\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}\sum_{i=j+1}^{n}\frac{1}{s_i^2-\eta_2}}{\Delta}$$

Substituting (8), (9) and (10) into $\mathbf{w} = \frac{\mathbf{\Omega}^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}\mathbf{\Omega}^{-1}\mathbf{1}}$ gives the weight of stock i in the GMVP:

$$w_i = \frac{1}{s_i^2-\eta_1}\cdot\frac{1+\left(\eta_2-\eta\right)\sum_{i=j+1}^{n}\frac{1}{s_i^2-\eta_2}}{\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}+\sum_{i=j+1}^{n}\frac{1}{s_i^2-\eta_2}+\left(\eta_1+\eta_2-2\eta\right)\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}\sum_{i=j+1}^{n}\frac{1}{s_i^2-\eta_2}} \quad , i=1,\ldots,j$$

(11)

$$w_i = \frac{1}{s_i^2-\eta_2}\cdot\frac{1+\left(\eta_1-\eta\right)\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}}{\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}+\sum_{i=j+1}^{n}\frac{1}{s_i^2-\eta_2}+\left(\eta_1+\eta_2-2\eta\right)\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}\sum_{i=j+1}^{n}\frac{1}{s_i^2-\eta_2}} \quad , i=j+1,\ldots,n$$

And now we can easily see that the vector $\mathbf{w}$ is strictly positive if the following set of conditions holds:

$$(12) \qquad \begin{aligned} &\eta_1 < \min\left(s_i^2\right) \ , \ i=1,\ldots,j \\ &\eta_2 < \min\left(s_i^2\right) \ , \ i=j+1,\ldots,n \\ &\eta \le \min\left(\eta_1,\eta_2\right) \end{aligned}$$

Phase 2—correlations not greater than |1|

We require the correlation coefficient between any pair of stocks k and l not to be greater than |1|. Therefore, we obtain also the following set of conditions on η_1, η_2 and η:

$$\begin{aligned} &-s_k s_l \le \eta_1 \le s_k s_l \quad \forall k,l \in \textit{first } j \textit{ stocks} \\ &-s_k s_l \le \eta_2 \le s_k s_l \quad \forall k,l \in \textit{last n-j stocks} \\ &-s_k s_l \le \eta \le s_k s_l \quad \forall k \in \textit{first } j \textit{ stocks} \ , \ l \in \textit{last n-j stocks} \end{aligned}$$

And base on this set of conditions and the set from (12), we generate the following set of sufficient conditions, for which the GMVP is strictly positive and the correlations are not greater than |1|:

$$
(13)\qquad
\begin{aligned}
&-\min\left(s_i^2\right) \le \eta_1 < \min\left(s_i^2\right) \ , \ i=1,\dots,j \\
&-\min\left(s_i^2\right) \le \eta_2 < \min\left(s_i^2\right) \ , \ i=j+1,\dots,n \\
&-\min_{i=1,\dots,j}\left(s_i\right)\cdot \min_{i=j+1,\dots,n}\left(s_i\right) \le \min\left(\eta_1,\eta_2\right) \\
&-\min_{i=1,\dots,j}\left(s_i\right)\cdot \min_{i=j+1,\dots,n}\left(s_i\right) \le \eta \le \min\left(\eta_1,\eta_2\right)
\end{aligned}
$$

Phase 3—strictly positive eigenvalues

$\mathbf{\Omega}$ is a symmetric real matrix. Therefore, it has real eigenvalues and it can be diagonalized:

$$
\mathbf{R}^{-1}\mathbf{\Omega}\mathbf{R} = \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ 0 & & \ddots & \\ & & & \lambda_n \end{pmatrix}
$$

where $\mathbf{R}$ denotes the diagonalizing matrix, $\mathbf{R}^{-1}$ denotes the inverse matrix of $\mathbf{R}$ and $\mathbf{\Lambda}$ denotes the diagonal matrix, whose diagonal elements $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $\mathbf{\Omega}$.

Hence, $\mathbf{\Omega R} = \mathbf{R\Lambda}$ and it is straightforward to see that the element *ij* of $\mathbf{\Omega R}$ is given by:

$$
\left(\mathbf{\Omega R}\right)_{ij} = \sum_{k=1}^{n} \mathbf{\Omega}_{ik}\mathbf{R}_{kj} = \lambda_j \mathbf{R}_{ij}
$$

We obtain here a set of equations that determines the elements of column j in $\mathbf{R}$. This is true for any column in $\mathbf{R}$, and for future convenient we omit the index j from the above expression. Therefore, we have:

$$\sum_{k=1}^{n}\mathbf{\Omega}_{ik}\mathbf{R}_k=\lambda\mathbf{R}_i$$

Now, substituting the elements of $\mathbf{\Omega}$ into our expression gives:

$$s_i^2\mathbf{R}_i+\eta_1\sum_{k=1,k\neq i}^{j}\mathbf{R}_k+\eta\sum_{k=j+1}^{n}\mathbf{R}_k=\lambda\mathbf{R}_i \quad , \quad i=1,\dots,j$$
$$s_i^2\mathbf{R}_i+\eta\sum_{k=1}^{j}\mathbf{R}_k+\eta_2\sum_{k=j+1,k\neq i}^{n}\mathbf{R}_k=\lambda\mathbf{R}_i \quad , \quad i=j+1,\dots,n$$

And after rearranging we obtain[1]:

$$\mathbf{R}_i=\frac{\eta_1}{\lambda-(s_i^2-\eta_1)}\sum_{k=1}^{j}\mathbf{R}_k+\frac{\eta}{\lambda-(s_i^2-\eta_1)}\sum_{k=j+1}^{n}\mathbf{R}_k=0 \quad , \quad i=1,\dots,j$$
$$\mathbf{R}_i=\frac{\eta}{\lambda-(s_i^2-\eta_2)}\sum_{k=1}^{j}\mathbf{R}_k+\frac{\eta_2}{\lambda-(s_i^2-\eta_2)}\sum_{k=j+1}^{n}\mathbf{R}_k=0 \quad , \quad i=j+1,\dots,n$$

Summing the two expressions over all the possible values of *i* (and replacing the index *i* with k) give:

$$\sum_{k=1}^{j}\mathbf{R}_k=\eta_1\sum_{k=1}^{j}\frac{1}{\lambda-(s_k^2-\eta_1)}\sum_{k=1}^{j}\mathbf{R}_k+\eta\sum_{k=1}^{j}\frac{1}{\lambda-(s_k^2-\eta_1)}\sum_{k=j+1}^{n}\mathbf{R}_k$$
$$\sum_{k=j+1}^{n}\mathbf{R}_k=\eta\sum_{k=j+1}^{n}\frac{1}{\lambda-(s_k^2-\eta_2)}\sum_{k=1}^{j}\mathbf{R}_k+\eta_2\sum_{k=j+1}^{n}\frac{1}{\lambda-(s_k^2-\eta_2)}\sum_{k=j+1}^{n}\mathbf{R}_k$$

And again after rearranging we obtain:

$$\left[\eta_1\sum_{k=1}^{j}\frac{1}{\lambda-(s_k^2-\eta_1)}-1\right]\sum_{k=1}^{j}\mathbf{R}_k+\eta\sum_{k=1}^{j}\frac{1}{\lambda-(s_k^2-\eta_1)}\sum_{k=j+1}^{n}\mathbf{R}_k=0$$
$$\eta\sum_{k=j+1}^{n}\frac{1}{\lambda-(s_k^2-\eta_2)}\sum_{k=1}^{j}\mathbf{R}_k+\left[\eta_2\sum_{k=j+1}^{n}\frac{1}{\lambda-(s_k^2-\eta_2)}-1\right]\sum_{k=j+1}^{n}\mathbf{R}_k=0$$

---

[1] $\mathbf{R}_i$ exists $\forall i$ and therefore there is no problem to divide the expressions by $\frac{1}{\lambda-(s_i^2-\eta_1)}$ and $\frac{1}{\lambda-(s_i^2-\eta_2)}$.

We obtain here two homogenous linear equations in two unknowns, $\sum_{k=1}^{j}\mathbf{R}_k$ and $\sum_{k=j+1}^{n}\mathbf{R}_k$. It can be shown that $\sum_{k=1}^{j}\mathbf{R}_k$ and $\sum_{k=j+1}^{n}\mathbf{R}_k$ cannot both equal zero. Recall that a set of homogenous linear equations has a solution other than the zero solution, if and only if its determinant equals 0. Therefore, in our case, as we know that $\sum_{k=1}^{j}\mathbf{R}_k$ and $\sum_{k=j+1}^{n}\mathbf{R}_k$ exist[2], we must obtain:

$$\left[\eta_1\sum_{k=1}^{j}\frac{1}{\lambda-(s_k^2-\eta_1)}-1\right]\left[\eta_2\sum_{k=j+1}^{n}\frac{1}{\lambda-(s_k^2-\eta_2)}-1\right]=\eta^2\sum_{k=1}^{j}\frac{1}{\lambda-(s_k^2-\eta_1)}\sum_{k=j+1}^{n}\frac{1}{\lambda-(s_k^2-\eta_2)}$$

Denote:

$$\mathrm{F}_1(\lambda)=\sum_{k=1}^{j}\frac{1}{\lambda-(s_k^2-\eta_1)} \quad \text{and} \quad \mathrm{F}_2(\lambda)=\sum_{k=j+1}^{n}\frac{1}{\lambda-(s_k^2-\eta_2)}$$

Therefore, our equation becomes:

$$\left[\eta_1\mathrm{F}_1(\lambda)-1\right]\left[\eta_2\mathrm{F}_2(\lambda)-1\right]=\eta^2\mathrm{F}_1(\lambda)\mathrm{F}_2(\lambda)$$

We are dealing with a finite $\mathbf{\Omega}$. Thus, $\lambda,\ s_k^2\ \forall k,\ \eta_1$ and η_2 are finite and therefore $\mathrm{F}_i(\lambda)\neq 0$, $i=1,2$. Hence, we can divide our expression by $\mathrm{F}_1(\lambda)\mathrm{F}_2(\lambda)$ and obtain:

$$(14)\qquad \left[\eta_1-\frac{1}{\mathrm{F}_1(\lambda)}\right]\left[\eta_2-\frac{1}{\mathrm{F}_2(\lambda)}\right]=\eta^2$$

[2] $\sum_{k=1}^{j}\mathbf{R}_k$ and $\sum_{k=j+1}^{n}\mathbf{R}_k$ exist because the diagonalizing procedure is well defined here.

It can be shown that for $\lambda \le 0$:

$$-\infty < \frac{1}{\mathrm{F}_i(\lambda)} \le -\frac{1}{\sum_i \frac{1}{s_k^2 - \eta_i}} \quad , \; i = 1,2$$

$$\text{where } \Sigma_i = \begin{cases} \sum_{k=1}^{j} & , \; i = 1 \\ \sum_{k=j+1}^{n} & , \; i = 2 \end{cases}$$

And therefore:

$$(15) \qquad \infty > \eta_i - \frac{1}{\mathrm{F}_i(\lambda)} \ge \eta_i + \frac{1}{\sum_i \frac{1}{s_k^2 - \eta_i}} \quad , \; i = 1,2 \;\; , \;\; \lambda \le 0$$

Now let us assume that:

$$(16) \qquad \eta_1 > -\frac{1}{\sum_{k=1}^{j} \frac{1}{s_k^2 - \eta_1}} \quad , \quad \eta_2 > -\frac{1}{\sum_{k=j+1}^{n} \frac{1}{s_k^2 - \eta_2}}$$

Thus, from (15) and (16) we obtain that:

$$\left[\eta_1 - \frac{1}{\mathrm{F}_1(\lambda)}\right]\left[\eta_2 - \frac{1}{\mathrm{F}_2(\lambda)}\right] \ge \left(\eta_1 + \frac{1}{\sum_{k=1}^{j} \frac{1}{s_k^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\sum_{k=j+1}^{n} \frac{1}{s_k^2 - \eta_2}}\right) \quad , \; \lambda \le 0$$

And if we also assume:

$$(17) \qquad \left(\eta_1 + \frac{1}{\sum_{k=1}^{j} \frac{1}{s_k^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\sum_{k=j+1}^{n} \frac{1}{s_k^2 - \eta_2}}\right) > \eta^2$$

Then:

$$\left[\eta_1 - \frac{1}{\mathrm{F}_1(\lambda)}\right]\left[\eta_2 - \frac{1}{\mathrm{F}_2(\lambda)}\right] > \eta^2 \quad , \; \lambda \le 0$$

Which means that under the conditions in (16) and (17) there are no non-positive eigenvalues for which equation (14) holds. However, we know that equation (14) must hold. Hence, we can say that under the conditions in (16) and (17), equation (14) holds only for strictly positive eigenvalues. In other words, we managed to find sufficient conditions [those in (16) and (17)], for which $\mathbf{\Omega}$ is positive definite[3].

Because of (16), we can denote:

$$\left|\eta_{12}^*\right| = +\sqrt{\left(\eta_1 + \frac{1}{\sum_{k=1}^{j}\frac{1}{s_k^2-\eta_1}}\right)\left(\eta_2 + \frac{1}{\sum_{k=j+1}^{n}\frac{1}{s_k^2-\eta_2}}\right)}$$

And now we can write again our set of sufficient conditions as follows:

$$\eta_1 > -\frac{1}{\sum_{k=1}^{j}\frac{1}{s_k^2-\eta_1}} \quad , \quad \eta_2 > -\frac{1}{\sum_{k=j+1}^{n}\frac{1}{s_k^2-\eta_2}} \quad , \quad -\left|\eta_{12}^*\right| < \eta < \left|\eta_{12}^*\right|$$

It can be shown that:

$$\eta_1 > -\frac{1}{\sum_{k=1}^{j}\frac{1}{s_k^2-\eta_1}} \quad iff \quad -\left|\eta_1^*\right| < \eta_1 \le \min\left(s_k^2\right) \quad , \quad k = 1,\ldots, j$$

$$\eta_2 > -\frac{1}{\sum_{k=j+1}^{n}\frac{1}{s_k^2-\eta_2}} \quad iff \quad -\left|\eta_2^*\right| < \eta_2 \le \min\left(s_k^2\right) \quad , \quad k = j+1,\ldots, n$$

where $\left|\eta_1^*\right|$ and $\left|\eta_2^*\right|$ are respectively the unique solutions of the following equations:

$$\left|\eta_1^*\right| = 1\Big/\sum_{k=1}^{j}\frac{1}{s_k^2+\left|\eta_1^*\right|} \quad \text{and} \quad \left|\eta_2^*\right| = 1\Big/\sum_{k=j+1}^{n}\frac{1}{s_k^2+\left|\eta_2^*\right|}$$

---

[3] It can also be shown that the sufficient conditions found are also necessary. However, we do not need it here, as our goal is to find a set of sufficient conditions on η_1, η_2 and η.

To sum up, $\mathbf{\Omega}$ is positive definite if the following set of conditions holds:

$$
(18)\qquad
\begin{aligned}
&-\left|\eta_1^*\right| < \eta_1 \le \min\left(s_i^2\right) \quad , \quad i = 1,\dots, j\\
&-\left|\eta_2^*\right| < \eta_2 \le \min\left(s_i^2\right) \quad , \quad i = j+1,\dots, n\\
&-\left|\eta_{12}^*\right| < \eta < \left|\eta_{12}^*\right|
\end{aligned}
$$

where $\left|\eta_1^*\right|$ and $\left|\eta_2^*\right|$ are respectively the unique solutions of the following equations:

$$
\left|\eta_1^*\right| = 1 \Big/ \sum_{i=1}^{j} \frac{1}{s_i^2 + \left|\eta_1^*\right|} \quad , \quad \left|\eta_2^*\right| = 1 \Big/ \sum_{i=j+1}^{n} \frac{1}{s_i^2 + \left|\eta_2^*\right|}
$$

and:

$$
\left|\eta_{12}^*\right| = +\sqrt{\left(\eta_1 + \frac{1}{\sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}\right)}
$$

For convenient, this time we replace the index *k* with *i*.

Now we can finish our proof. Combining the set of conditions from (13) and (18) gives the sufficient conditions that appear in the theorem:

$$
\begin{cases}
-\left|\eta_1^*\right| < \eta_1 < \min\left(s_i^2\right) & , \quad \left|\eta_1^*\right| < \min\left(s_i^2\right) \ , \ i = 1,\dots, j\\
-\min\left(s_i^2\right) \le \eta_1 < \min\left(s_i^2\right) & , \quad \left|\eta_1^*\right| \ge \min\left(s_i^2\right) \ , \ i = 1,\dots, j
\end{cases}
$$

$$
\begin{cases}
-\left|\eta_2^*\right| < \eta_2 < \min\left(s_i^2\right) & , \quad \left|\eta_2^*\right| < \min\left(s_i^2\right) \ , \ i = j+1,\dots, n\\
-\min\left(s_i^2\right) \le \eta_2 < \min\left(s_i^2\right) & , \quad \left|\eta_2^*\right| \ge \min\left(s_i^2\right) \ , \ i = j+1,\dots, n
\end{cases}
$$

$$
\begin{cases}
-\left|\eta_{12}^*\right| < \eta < \left|\eta_{12}^*\right| & , \quad \left|\eta_{12}^*\right| < \min\left(\eta_1, \eta_2\right)\\
-\min\limits_{i=1,\dots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\dots,n}\left(s_i\right) \le \eta \le \min\left(\eta_1, \eta_2\right) & , \quad \left|\eta_{12}^*\right| > \min\limits_{i=1,\dots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\dots,n}\left(s_i\right)\\
-\left|\eta_{12}^*\right| < \eta \le \min\left(\eta_1, \eta_2\right) & , \quad \min\left(\eta_1, \eta_2\right) < \left|\eta_{12}^*\right| \le \min\limits_{i=1,\dots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\dots,n}\left(s_i\right)
\end{cases}
$$

The proof of the corollary is straightforward. When η_1 and η_2 are restricted to be nonnegative, $\left|\eta_{12}^*\right|$ cannot be smaller than $\min\left(\eta_1, \eta_2\right)$, and therefore the simpler version of the sufficient conditions is obtained:

$$0 \le \eta_1 < \min\left(s_i^2\right) \quad , \quad i = 1, \ldots, j$$

$$0 \le \eta_2 < \min\left(s_i^2\right) \quad , \quad i = j+1, \ldots, n$$

$$\begin{cases} -\min\limits_{i=1,\ldots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \le \eta \le \min\left(\eta_1, \eta_2\right) & , \quad \left|\eta_{12}^*\right| > \min\limits_{i=1,\ldots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \\ -\left|\eta_{12}^*\right| < \eta \le \min\left(\eta_1, \eta_2\right) & , \quad \min\left(\eta_1, \eta_2\right) < \left|\eta_{12}^*\right| \le \min\limits_{i=1,\ldots,j}\left(s_i\right) \cdot \min\limits_{i=j+1,\ldots,n}\left(s_i\right) \end{cases}$$